Particle Motion in the Stable Region Near the Edge of a Linear Sum Resonance Stopband

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ABSTRACT

This paper studies the particle motion when the tune is in the stable region close to the edge of linear sum resonance stopband. Results are found for the tune and the beta functions. Results are also found for the two solutions of the equations of motion. The results found are shown to be also valid for small accelerators where the large accelerator approximation may not be used.

1. Introduction

This paper studies the motion of a particle whose tune is near an edge of a linear sum resonance stopband. It is assumed that the tune is not near any other linear resonance, and the motion is dominated by the linear sum resonance. It is assumed that the linear sum resonance is being driven by a skew quadrupole field perturbation. When the unperturbed tune ν_{x0}, ν_{y0} is close to the resonance line $\nu_x + \nu_y = q$, q being an integer, the particle motion can be unstable. The region of instability is called the stopband. Results are found for the tune and the beta functions when the unperturbed tune is in the stable region but close to an edge of the stopband. Results are also found for the two solutions of the equations of motion. All the results found are shown to be also valid for small accelerators where the large accelerator approximation may not be used.

2. Results When The Tune Is Inside The Stopband

It will be assumed that in the absence of the perturbing fields, the tune of the particle is given by ν_{x0}, ν_{y0}, the x and y motions are uncoupled, and that the motion is stable when ν_{x0}, ν_{y0} is close to the line $\nu_{x0} + \nu_{y0} = q$, where q is an integer. It is assumed that a perturbing field is then added which is given by the skew quadrupole field

$$\begin{aligned} \Delta B_x &= -B_0 a_1 \ x \\ \Delta B_y &= B_0 a_1 \ y \end{aligned} \tag{2-1}$$

a_1 is the skew quadrupole multipole and $a_1 = a_1(s)$. B_0 is some standard field, usually the field in the main dipoles of the lattice.

The coupled equations of motion can be written as

$$\begin{aligned} \frac{d^2\eta_x}{d\theta_x^2} + \nu_{x0}^2\eta_x &= f_x \\ \frac{d^2\eta_y}{d\theta_y^2} + \nu_{y0}^2\eta_y &= f_y \\ f_x &= \nu_{x0}^2\beta_x^{3/2}\Delta B_y/B\rho \\ f_y &= -\nu_{y0}^2\beta_y^{3/2}\Delta B_x/B\rho \\ \eta_x &= x/\beta_x^{\frac{1}{2}}, \quad \eta_y = y/\beta_y^{\frac{1}{2}} \\ \rho &= B\rho/B_0, \quad B\rho = pc/e \\ ds &= \nu_{x0}\beta_x d\theta_x = \nu_{y0}\beta_y d\theta_y \end{aligned} \tag{2-2}$$

β_x, β_y are the unperturbed beta functions.

Eqs. (2-2) are valid for large accelerators, and some changes are required [1] to make them valid for small accelerators (see section 6). However the final results found below are also valid for small accelerators that require the use of the exact linearized equations. This is shown in section 6.

Eq. (2-2) can be rewritten as

$$\begin{aligned} \frac{d^2}{d\theta_x^2}\eta_x + \nu_{x0}^2\eta_x &= b_x \ \eta_y \\ \frac{d^2}{d\theta_y^2}\eta_y + \nu_{y0}^2\eta_y &= b_y \ \eta_x \\ b_x &= -\nu_{x0}^2\beta_x\left(\beta_x\beta_y\right)^{\frac{1}{2}} a_1/\rho \\ b_y &= -\nu_{y0}^2\beta_y\left(\beta_x\beta_y\right)^{\frac{1}{2}} a_1/\rho \end{aligned} \tag{2-3}$$

Eqs. (2-3) are a set of linear equations for η_x, η_y with coefficients that are periodic in s. The extension of Floquet's theorem to more than one dimension applies, and the solutions have the Floquet form,

$$\begin{bmatrix} \eta_x \\ \eta_y \end{bmatrix} = \exp\left(i\nu_x\theta_x\right) \begin{bmatrix} h_x \\ h_y \end{bmatrix} \tag{2-4}$$

h_x and h_y are periodic in s. The solution can also have the form

$$\begin{bmatrix} \eta_x \\ \eta_y \end{bmatrix} = \exp\left(i\nu_y\theta_y\right) \begin{bmatrix} h_x \\ h_y \end{bmatrix} \tag{2-5}$$

If for small a_1 one finds a solution of the form Eq. (2-4) where $h_y \to 0$ when $a_1 \to 0$, then $\nu_x \to \nu_{x0}$. This solution reduces to the uncoupled x motion when $a_1 \to 0$ and will be called the ν_x mode. If for small a_1, one finds a solution of the form Eq. (2-5) where $h_x \to 0$, when $a_1 \to 0$, then $\nu_y \to \nu_{y0}$ and thus this solution will be called the ν_y mode.

To find the solution that corresponds to the ν_x mode, one can assume that η_x has the form

$$\begin{aligned} \eta_x &= A_s \exp\left(i\nu_{xs}\theta_x\right) + \sum_{r\neq s} A_r \exp\left(i\nu_{xr}\theta_x\right) \\ \nu_{xr} &= \nu_{xs} + n, \quad n \text{ an integer}, n \neq 0 \end{aligned} \tag{2-6a}$$

where for small enough a_1, $A_r \ll A_s$ and $\nu_{xs} \to \nu_{x0}$ for $a_1 \to 0$. For the corresponding form for η_y one might assume for η_y

$$\begin{aligned} \eta_y &= \sum_r B_r \exp\left(i\nu_{yr}\theta_y\right) \\ \nu_{yr} &= \nu_{xs} + n \end{aligned} \tag{2-6b}$$

where $B_r \ll A_s$ for small enough a_1.

Eqs. (2-6) have the form given by Eq. (2-4) for the ν_x mode. It will be seen below, that the solution assumed for η_y Eq. (2-6b) is valid if one is not near the sum resonance $\nu_x + \nu_y = q$, q being an integer. When ν_{x0}, ν_{y0} are close to the sum resonance $\nu_x + \nu_y = q$, then one of the B_r will become as large as A_s and this is the B_r for which $\nu_{yr} = \nu_{xs} - q$. This is shown below. Thus, one assumes for η_y the solution with the form

$$\begin{aligned} \eta_y &= B_{\overline{s}} \exp\left(i\nu_{y\overline{s}}\theta_y\right) + \sum_{r\neq \overline{s}} B_r \exp\left(i\nu_{yr}\theta_y\right) \\ \nu_{y\overline{s}} &= \nu_{xs} - q \\ \nu_{yr} &= \nu_{xs} + n, \quad n \neq -q \end{aligned} \tag{2-6c}$$

Here $B_r \ll A_s$ but $B_{\overline{s}} \simeq A_s$. It is being assumed that ν_{x0}, ν_{y0} are not close to any other resonance other than $\nu_x + \nu_y = q$.

Putting this assumed form for η_x, η_y into the differential equations Eq. (2-3), one gets the equations for A_r, B_r

$$\begin{aligned}
\left(\nu_{xr}^2 - \nu_{x0}^2\right) A_r &= -2\nu_{x0} \sum_{r'} b_x \left(-\nu_{xr}, \nu_{yr'}\right) B_{r'} \\
\left(\nu_{yr}^2 - \nu_{y0}^2\right) B_r &= -2\nu_{y0} \sum_{r'} b_y \left(-\nu_{yr}, \nu_{xr'}\right) A_{r'} \\
b_x \left(-\nu_{xr}, \nu_{yr'}\right) &= \frac{1}{4\pi} \int_0^L ds \left(\beta_x \beta_y\right)^{\frac{1}{2}} \left(a_1/\rho\right) \exp\left[i\left(-\nu_{xr}\theta_x + \nu_{yr'}\theta_y\right)\right] \\
b_y \left(-\nu_{yr}, \nu_{xr'}\right) &= \frac{1}{4\pi} \int_0^L ds \left(\beta_x \beta_y\right)^{\frac{1}{2}} \left(a_1/\rho\right) \exp\left[i\left(-\nu_{yr}\theta_y + \nu_{xr'}\theta_x\right)\right]
\end{aligned} \tag{2-7}$$

L is the lattice circumference.

Eqs. (2-7) can be solved by an iterative perturbation procedure. For the initial guess for η_x, η_y in the iterative procedure one can assume

$$\begin{aligned}
\eta_x &= A_s \exp\left(i\nu_x\theta_x\right) \\
\eta_y &= B_{\overline{s}} \exp\left(i\nu_{y\overline{s}}\theta_y\right) \\
\nu_{y\overline{s}} &= \nu_{xs} - q = -\left(q - \nu_{xs}\right)
\end{aligned} \tag{2-8}$$

One can put this initial guess for η_x, η_y in the right hand side of Eq. (2-7) and solve for the A_r, B_r which gives

$$\begin{aligned}
\left(\nu_{xr}^2 - \nu_{x0}^2\right) A_r &= -2\nu_{x0} b_x \left(-\nu_{xr}, \nu_{y\overline{s}}\right) B_{\overline{s}} \\
\left(\nu_{yr}^2 - \nu_{y0}^2\right) B_r &= -2\nu_{y0} b_y \left(-\nu_{yr}, \nu_{xs}\right) A_s \\
\nu_{xr} = \nu_{xs} + n, &\quad \nu_{yr} = \nu_{xs} + m
\end{aligned} \tag{2-9}$$

For $A_r = A_s$ and $B_r = B_{\overline{s}}$ one finds

$$\begin{aligned}
\left(\nu_{xs}^2 - \nu_{x0}^2\right) A_s &= -2\nu_{x0} b_x \left(-\nu_{xs}, \nu_{y\overline{s}}\right) B_{\overline{s}} \\
\left(\nu_{y\overline{s}}^2 - \nu_{y0}^2\right) B_{\overline{s}} &= -2\nu_{y0} b_x \left(-\nu_{y\overline{s}}, \nu_{xs}\right) A_s \\
\nu_{y\overline{s}} &= -\left(q - \nu_{xs}\right)
\end{aligned} \tag{2-10}$$

Eqs. (2-10) are two linear homogeneous equations for A_s and $B_{\overline{s}}$, and to be solvable, we must have

$$\begin{aligned}
\left(\nu_{xs}^2 - \nu_{x0}^2\right)\left(\nu_{y\overline{s}}^2 - \nu_{y0}^2\right) &= 4\nu_{x0}\nu_{y0} |\Delta\nu_x|^2 \\
\Delta\nu_x &= \frac{1}{4\pi} \int_0^L ds \left(\beta_x \beta_y\right)^{\frac{1}{2}} \left(a_1/\rho\right) \exp\left[-i\left(\nu_{x0}\theta_x + \left(q - \nu_{x0}\right)\theta_y\right)\right]
\end{aligned} \tag{2-11}$$

where one uses $b_y(-\nu_{y\bar{s}}, \nu_{xs}) = b_x^*(-\nu_{xs}, \nu_{y\bar{s}})$. Eq. (2-11) is an equation for ν_{xs}, the tune of the ν_x mode, which is the mode where $\nu_{xs} \to \nu_{x0}$ when $a_1 \to 0$. It will be assumed that ν_{x0}, ν_{y0} is close to the resonance line $\nu_x + \nu_y = q$ and one can write

$$\begin{aligned} \left(\nu_{xs}^2 - \nu_{x0}^2\right) &= \left(\nu_{xs} + \nu_{x0}\right)\left(\nu_{xs} - \nu_{x0}\right) \simeq 2\nu_{x0}\left(\nu_{xs} - \nu_{x0}\right) \\ \left(\nu_{y\bar{s}}^2 - \nu_{y0}^2\right) &= \left(\nu_{y\bar{s}} + \nu_{y0}\right)\left(\left|\nu_{y\bar{s}}\right| - \nu_{y0}\right) \simeq 2\nu_{y0}\left(q - \nu_{xs} - \nu_{y0}\right) \end{aligned} \tag{2-12}$$

Eq. (2-11) then becomes

$$\left(\nu_{xs} - \nu_{x0}\right)\left(q - \nu_{xs} - \nu_{y0}\right) = \left|\Delta\nu_x\right| \tag{2-13}$$

To solve Eq. (2-13) one puts

$$\nu_{xs} = \nu_{xsR} - ig_x$$

where ν_{xsR} and g_x are both real, which gives the equation

$$\left(\nu_{xsR} - ig_x - \nu_{x0}\right)\left(q - \nu_{xsR} + ig_x - \nu_{y0}\right) = \left|\Delta\nu_x\right|^2 \tag{2-14}$$

The imagninary part of Eq. (2-14) gives

$$g_x\left[\nu_{xsR} - \nu_{x0}\left(q - \nu_{xsR} - \nu_{y0}\right)\right] = 0 \tag{2-15}$$

If one is inside the stopband, then $g_x \neq 0$ and one gets

$$\nu_{xsR} = \frac{1}{2}\left[\nu_{x0} + q - \nu_{y0}\right] \tag{2-16}$$

The real part of Eq. (2-14) gives

$$\left(\nu_{xsR} - \nu_{x0}\right)\left(q - \nu_{xsR} - \nu_{y0}\right) + q_x^2 = \left|\Delta\nu_x\right|^2 \tag{2-17}$$

Using Eq. (2-16) for ν_{xsR} one has

$$\begin{aligned} \nu_{xsR} - \nu_{x0} &= \frac{1}{2}\left[-\nu_{x0} + \left(q - \nu_{y0}\right)\right] \\ q - \nu_{xsR} - \nu_{y0} &= \frac{1}{2}\left[\left(q - \nu_{y0}\right) - \nu_{x0}\right] \end{aligned} \tag{2-18}$$

Eq. (2-17) then gives

$$\begin{aligned} g_x^2 + \left[\frac{1}{2}\left(q - \nu_{x0} - \nu_{y0}\right)\right]^2 &= \left|\Delta\nu_x\right|^2 \\ g_x = \pm\left\{\left|\Delta\nu_x\right|^2 - \left[\frac{1}{2}\left(q - \nu_{x0} - \nu_{y0}\right)\right]^2\right\}^{1/2} & \end{aligned} \tag{2-19}$$

Eq. (2-19) shows that the growth factor g_x has a maximum value of $g_x = |\Delta\nu_x|$ when ν_{x0}, ν_{y0} are on the resonance line, $q - \nu_{x0} - \nu_{y0} = 0$, and then decreases to zero at the edges of the stopband given by the two lines

$$q - \nu_{x0} - \nu_{y0} = \pm 2|\Delta\nu_x| \tag{2-20}$$

Eq. (2-19) shows that the unstable region in ν_{x0}, ν_{y0} is bounded by the two lines given by Eq. (2-20). These two lines are parallel to the resonance line $q - \nu_{x0} - \nu_{y0} = 0$, which lies midway between these two lines. If one wanted to define a stopband width, one might define it as the distance in ν_{x0}, ν_{y0} space across the unstable region, along a path which is perpendicular to the two boundary lines. This is given by

$$\text{stopband width} = 2.828|\Delta\nu_x| \tag{2-21}$$

For particle motion in 2 dimensional phase space, it has been found [2] that the real part of the tune is constant as the unperturbed tune moves across the stopband. This is not in general true for 4 dimensional phase, as the real part of the tune, given by Eq. (2-16), depends on the path in ν_{x0}, ν_{y0} which is chosen in crossing the stopband. However, if one chooses a path which is perpendicular to the resonance line, $q - \nu_{x0} - \nu_{y0} = 0$, then the real part of the tune does remain constant. One can see this by observing that if starting from ν_{x0}, ν_{y0} one draws a line perpendicular to the resonance line, the point on the resonance line that this perpendicular meets has the coordinates

$$\frac{1}{2}\left(\nu_{x0} + q - \nu_{y0}\right), \quad \frac{1}{2}\left(\nu_{y0} + q - \nu_{x0}\right) \tag{2-22}$$

The ν_x coordinate of this point is just ν_{xsR} as given by Eq. (2-16).

The above results are for the ν_x mode, the mode for which the tune approaches ν_{x0} when $a_1 \to 0$. One can find the corresponding results for the ν_y mode by using the following substitutions

$$\begin{aligned} \nu_{x0} &\to \nu_{y0} \\ \nu_{y0} &\to \nu_{x0} \\ \Delta\nu_x &\to \Delta\nu_y \\ g_x &\to g_y \end{aligned} \tag{2-23}$$

The apparent differences between g_x and g_y and $\Delta\nu_x$ and $\Delta\nu_y$ are negligible if the ν_{x0}, ν_{y0} are close to the resonance line. $\Delta\nu_x$, $\Delta\nu_y$ can be written as

$$\Delta\nu = \frac{1}{4\pi}\int ds\,(\beta_x\beta_y)^{\frac{1}{2}}\,(a_1/\rho)\exp\left[-i\,(\nu_x\theta_x+\nu_y\theta_y)\right] \tag{2-24a}$$

where for the ν_x mode, ν_x, ν_y is the point on the resonance line

$$\nu_x = \nu_{x0}, \nu_y = q - \nu_{x0} \tag{2-24b}$$

and for the ν_y mode, ν_x, ν_y is the point on the resonance line.

$$\nu_x = q - \nu_{y0}, \nu_y = \nu_{y0} \tag{2-24c}$$

These two points on the resonance line are close if ν_{x0}, ν_{y0} is assumed to be close to the resonance line, and their difference can be neglected. A reasonable compromise might be to choose for ν_x, ν_y the point on the resonance line which is midway between these two points, which is the choice of ν_x, ν_y given by

$$\nu_x = \frac{1}{2}\left(\nu_{x0}+q-\nu_{y0}\right), \nu_y = \frac{1}{2}\left(\nu_{y0}+q-\nu_{x0}\right) \tag{2-25}$$

3. Solutions of the Equations of Motion

Now let us find the solutions for η_x, η_y that will give the particle motion inside the stopband. To lowest order η_x and η_y are given by Eqs. (2-6) for the ν_x mode as

$$\begin{aligned}\eta_x &= A_s\exp\left(i\nu_{xs}\theta_x\right)\\ \eta_y &= B_{\overline{s}}\exp\left(i\nu_{y\overline{s}}\theta_y\right)\\ \nu_{y\overline{s}} &= \nu_{xs}-q\end{aligned} \tag{3-1}$$

From Eq. (2-10) one finds that inside the stopband

$$B_{\overline{s}} = -\frac{\Delta\nu_x^*}{|\nu_{y\overline{s}}|-\nu_{y0}}A_s \tag{3-2}$$

Since

$$\begin{aligned}|\nu_{y\overline{s}}|-\nu_{y0} &= q-\nu_{xs}-\nu_{y0}\\ &= q-\nu_{xsR}+ig_x-\nu_{y0}\\ &= q-\nu_{y0}-\frac{1}{2}\left[\nu_{x0}+q-\nu_{y0}\right]+ig_x\\ &= \frac{1}{2}\left(q-\nu_{x0}-\nu_{y0}\right)+ig_x\end{aligned} \tag{3-3}$$

one gets

$$
\begin{aligned}
B_{\overline{s}} &= -\exp\left[-i\left(\delta_{1x}+\delta_{2x}\right)\right] A_s \\
\delta_{1x} &= ph\left(\Delta\nu_x\right) \\
\delta_{2x} &= ph\left[\frac{1}{2}\left(q-\nu_{x0}-\nu_{y0}\right)+ig_x\right]
\end{aligned}
\tag{3-4}
$$

where use was made of Eq. (2-19) and ph indicates the phase of a variable.

Thus, inside the stopband, η_x and η_y are of the same order of magnitude one can find the first order correction to η_x and η_y using Eq. (2-9). The results for η_x and η_y for the ν_y mode can be found by using the substitutions given by Eqs. (2-23).

Results will now be found for η_x and η_y which are correct to first order in the perturbation and when ν_{x0}, ν_{y0} is inside the stopband or in the stable region near an edge of the stopband. η_x, η_y are given by Eqs. (2-6). For the ν_x mode

$$
\begin{aligned}
\eta_x &= A_s\exp\left(i\nu_{xs}\theta_x\right)+\sum_{r\neq s} A_r\exp\left(i\nu_{xr}\theta_x\right) \\
\eta_y &= B_{\overline{s}}\exp\left(i\nu_{y\overline{s}}\theta_y\right)+\sum_{r\neq s} B_r\exp\left(i\nu_{yr}\theta_y\right) \\
\nu_{y\overline{s}} &= \nu_{xs}-q \\
\nu_{yr} &= \nu_{xs}+n, \quad n\neq -q \\
\nu_{xr} &= \nu_{xs}+n, \quad n\neq 0
\end{aligned}
\tag{3-5}
$$

The first order solution is given by Eqs. (2-9) and (2-10). $B_{\overline{s}}$, A_r and B_r are given by Eq. (2-7)

$$
\begin{aligned}
B_{\overline{s}} &= -2\nu_{y0}b_y\left(-\nu_{y\overline{s}},\nu_{xs}\right)A_s/\left(\nu_{y\overline{s}}^2-\nu_{y0}^2\right) \\
A_r &= -2\nu_{x0}b_x\left(-\nu_{xr},\nu_{y\overline{s}}\right)B_{\overline{s}}/\left(\nu_{xr}^2-\nu_{x0}^2\right) \\
B_r &= -2\nu_{y0}b_y\left(-\nu_{yr},\nu_{xs}\right)A_s/\left(\nu_{yr}^2-\nu_{y0}^2\right) \\
\nu_{xr} &= \nu_{xs}-m \quad m\neq 0 \\
\nu_{yr} &= \nu_{y\overline{s}}+n \quad n\neq 0
\end{aligned}
\tag{3-6}
$$

First, let us compute $B_{\overline{s}}$.

$$
\begin{aligned}
\nu_{y\overline{s}}^2-\nu_{y0}^2 &= \left(\left|\nu_{y\overline{s}}\right|+\nu_{y0}\right)\left(\left|\nu_{y\overline{s}}\right|-\nu_{y0}\right) \\
&= 2\nu_{y0}\left(q-\nu_{xs}-\nu_{y0}\right)
\end{aligned}
\tag{3-7}
$$

We can write for ν_{xs}

$$
\nu_{xs}=\frac{1}{2}\left(\nu_{x0}+q-\nu_{y0}\right)-\delta_x \tag{3-8}
$$

where $\delta_x = ig_x$, see Eq. (2-19) for g_x, when the tune is inside the stopband. In the stable region near an edge of a stopband, ν_{xs} and δ_x are given in section 4. Note that $\delta_x = 0$ when the tune is on the edge of the stopband. Then Eq. (3-7) gives

$$\nu_{y\bar{s}}^2 - \nu_{y0}^2 = 2\nu_{y0}\left(\frac{1}{2}\left(q - \nu_{x0} - \nu_{y0}\right) + \delta_x\right) \tag{3-9}$$

One also finds

$$\begin{aligned} b_x\left(-\nu_{y\bar{s}}, \nu_{xs}\right) &= \Delta\nu_x^* \\ B_{\bar{s}} &= -d_x \exp\left(-i\delta_{1x}\right) A_s \\ d_x &= -\frac{|\Delta\nu_x|}{\frac{1}{2}\left(q - \nu_{x0} - \nu_{y0}\right) + \delta_x} \exp\left(-i\delta_{1x}\right) A_s \\ \delta_{1x} &= ph\left(\Delta\nu_x\right) \end{aligned} \tag{3-10}$$

One may note that inside the stopband

$$\begin{aligned} d_x &= -\exp\left[-i\left(\delta_{1x} + \delta_{2x}\right)\right] A_s \\ \delta_{2x} &= ph\left[\frac{1}{2}\left(q - \nu_{x0} - \nu_{y0}\right) + ig_x\right] \end{aligned} \tag{3-11}$$

One may now find the A_r from Eq. (3-6)

$$\begin{aligned} \nu_{xr} &= \nu_{xs} - m, \qquad m \neq 0 \\ \nu_{xr}^2 - \nu_{x0}^2 &= m\left(m - 2\nu_{x0}\right) \\ b_x\left(-\nu_{xr}, \nu_{y\bar{s}}\right) &= b_m \\ b_m &= \frac{1}{4\pi}\int ds\left(\frac{a_1}{\rho}\right)\left(\beta_x\beta_y\right)^{\frac{1}{2}} \exp\left[-i\left(\left(q - \nu_{x0}\right)\theta_y + \nu_{x0}\theta_x\right) + im\theta_x\right] \\ A_r &= \frac{-2\nu_{x0}}{m\left(m - 2\nu_{x0}\right)} b_m B_{\bar{s}} \\ A_r &= \frac{-2\nu_{x0}}{m\left(m - 2\nu_{x0}\right)} d_x b_m \exp\left(-i\delta_{1x}\right) A_s \end{aligned} \tag{3-12}$$

One may find the B_r from Eq. (3-6)

$$\begin{aligned} \nu_{yr} &= \nu_{y\bar{s}} + n, \qquad n = 0 \\ \nu_{y\bar{s}} &= \nu_{xs} - q \\ \nu_{yr}^2 - \nu_{y0}^2 &= n\left(n - 2\left(q - \nu_{x0}\right)\right) \\ b_y\left(-\nu_{xr}, \nu_{xs}\right) &= c_n^* \\ c_n &= \frac{1}{4\pi}\int ds\left(\frac{a_1}{\rho}\right)\left(\beta_x\beta_y\right)^{\frac{1}{2}} \exp\left[-i\left(\left(q - \nu_{x0}\right)\theta_y + \nu_{x0}\theta_x\right) + in\theta_y\right] \\ B_r &= \frac{-2\nu_{y0}}{n\left(n - 2\left(q - \nu_{x0}\right)\right)} c_m^*\, A_s \end{aligned} \tag{3-13}$$

Putting these results for $B_{\bar{s}}$, A_r, B_r into Eq. (3-5), one finds the following results for η_x, η_y for the ν_x mode.

$$\begin{aligned}
\eta_x &= A_s \exp\left(i\nu_{xs}\theta_x\right)\left[1+\sum_{m\neq 0} f_m \exp\left(-im\theta_x\right)\right] \\
f_m &= \frac{-2\nu_{x0}}{m\left(m-2\nu_{x0}\right)} d_x b_m \exp\left(-i\delta_{1x}\right) \\
d_x &= \frac{-|\Delta\nu_x|}{\frac{1}{2}\left(q-\nu_{y0}-\nu_{x0}\right)+\delta_x} \\
\eta_y &= A_s \exp\left(i\nu_{y\bar{s}}\theta_y\right)\left[1+\sum_{n\neq 0} g_n \exp\left(in\theta_y\right)\right] \\
g_n &= \frac{-2\nu_{y0}}{n\left(n-2\left(q-\nu_{x0}\right)\right)} C_n^* \\
\nu_{y\bar{s}} &= \nu_{xs} - q
\end{aligned} \tag{3-14}$$

δ_x is given by $i\ g_x$ inside the stopband where g_x is given by Eq. (2-19) as

$$g_x = \left\{ |\Delta\nu_x|^2 - \left[\frac{1}{2}\left(q-\nu_{x0}-\nu_{y0}\right)\right]^2 \right\}^{\frac{1}{2}} \tag{3-15a}$$

In the stable region near a stopband edge, δ_x is given by Eq. (4-10) as

$$\begin{aligned}
|\delta_x| &= \left\{\epsilon_x\left(|\Delta\nu_x| + \epsilon_x/4\right)\right\}^{\frac{1}{2}} \\
\epsilon_x &= |q \pm |2\Delta\nu_x| - \nu_{x0} - \nu_{y0}|
\end{aligned} \tag{3-15b}$$

One uses the + sign for the upper stopband edge and the − sign for the lower edge. δ_x is positive for the lower edge and negative for the upper edge.

Equations (3-4) give the solutions of the equations of motion for the ν_x mode. The solutions for the ν_y mode are found by replacing each parameter for the ν_x mode by its corresponding parameter for the ν_y mode.

One may note from Eq. (3-14) that for the ν_x mode the dominant harmonic for η_x is $m \simeq 2\nu_{x0}$, and for η_y $n = 2|\nu_{y\bar{s}}| = 2(q-\nu_{x0})$. Keeping just the dominant harmonics give fairly simple results for η_x, η_y.

4. The Tune Near the Edge of a Stopband

In this section, a result will be found for the tune in the stable region outside the stopband but close to an edge of the stopband. It will be shown that close to an edge of the stopband the tune of the ν_x mode is given by

$$\begin{aligned} |\nu_x - \frac{1}{2}(\nu_{x0} + q - \nu_{y0})| &= \{\epsilon_x |\Delta\nu_x|\}^{\frac{1}{2}} \\ \epsilon_x &= |q \pm 2|\Delta\nu_x| - \nu_{x0} - \nu_{y0}| \end{aligned} \tag{4-1}$$

ν_x is the tune of the ν_x mode, ϵ is the distance from ν_{x0}, ν_{y0} to the edge of the stopband. In the $\pm$, the $+$ sign is for the upper edge, and the $-$ sign for the lower edge. When ν_{x0}, ν_{y0} reaches the edge of the stopband, then $\epsilon = 0$, and $\nu_x = \frac{1}{2}(\nu_{x0} + q - \nu_{y0})$, which according to Eq. (2-16) is the real part of the tune inside the stopband.

Eq. (4-1) shows that near the stopband edge, ν_x varies rapidly with ϵ_x. As one reaches the edge of the stopband, ϵ_x goes to zero and $d\nu_x/d\epsilon_x$ becomes infinite like $\epsilon_x^{-\frac{1}{2}}$.

To find ν_x in the stable region outside the stopband, where $|q - \nu_{x0} - \nu_{y0}| > 2|\Delta\nu_x|$, one goes back to the derivation given in section 2 for ν_x inside the stopband, starting with Eq. (2-13)

$$(\nu_x - \nu_{x0})(q - \nu_x - \nu_{y0}) = |\Delta\nu_x|^2 \tag{4-2}$$

Because of the condition that ν_x is outside the stopband or

$$|q - \nu_{x0} - \nu_{y0}| > 2|\Delta\nu_x| \tag{4-3}$$

one sees that one must have $g_x = 0$, as Eq. (2-19) would indicate g_x is imaginary which contradicts the assumption that g_x is real.

Let us assume that we start with ν_{x0}, ν_{y0} below the lower stopband edge and let ν_{x0}, ν_{y0} approach the lower stopband edge. The equation of the lower stopband edge is given by

$$q - \nu_{x0} - \nu_{y0} = 2|\Delta\nu_x| \tag{4-4}$$

when ν_{x0}, ν_{y0} arrive on the lower stopband edge, then ν_x will arrive at the value $\nu_x = \frac{1}{2}(\nu_{x0} + q - \nu_{y0})$ as indicated by Eq. (2-16). Thus below the stopband edge one can write

$$\nu_x = \frac{1}{2}(\nu_{x0} + q - \nu_{y0}) - \delta_x \tag{4-5}$$

where $\delta_x \to 0$ when ν_{x0}, ν_{y0} arrive at the stopband edge. We then find

$$\begin{aligned} \nu_x - \nu_{x0} &= \frac{1}{2}\left(q - \nu_{x0} - \nu_{y0}\right) - \delta_x \\ q - \nu_x - \nu_{y0} &= \frac{1}{2}\left(q - \nu_{x0} - \nu_{y0}\right) + \delta_x \end{aligned} \tag{4-6}$$

and Eq. (3-2) becomes

$$\begin{aligned} \left[\frac{1}{2}\left(q - \nu_{x0} - \nu_{y0}\right)\right]^2 - \delta_x^2 &= |\Delta\nu_x|^2 \\ \delta_x &= \left\{\left[\frac{1}{2}\left(q - \nu_{x0} - \nu_{y0}\right)\right]^2 - |\Delta\nu_x|^2\right\}^{\frac{1}{2}} \end{aligned} \tag{4-7}$$

Eq. (4-7) gives ν_x in the stable region near the stopband. It can be put in another form that indicates the dependence on the distance from ν_{x0}, ν_{y0} to the stopband edge.

Below the stopband, one writes

$$\epsilon_x = q - 2|\Delta\nu_x| - \nu_{x0} - \nu_{y0} \tag{4-8}$$

where ϵ_x indicates the distance from ν_{x0}, ν_{y0} to the stopband edge which is given by Eq. (4-4). When ν_{x0}, ν_{y0} is on the stopband edge and $\nu_{x0} + \nu_{y0} = q - 2|\Delta\nu_x|$ then $\epsilon_x = 0$.

Using Eq. (4-8) to replace $q - \nu_{x0} - \nu_{y0}$ by $\epsilon_x + 2|\Delta\nu_x|$ in Eq. (3-7) one finds

$$\delta_x = \left\{\epsilon_x\left(|\Delta\nu_x| + \epsilon_x/4\right)\right\}^{\frac{1}{2}} \tag{4-9}$$

Eq. (4-9) can then be written so as to hold both above and below the stopband to give

$$\begin{aligned} \left|\nu_x - \frac{1}{2}\left(\nu_{x0} + q - \nu_{y0}\right)\right| &= \left\{\epsilon_x\left(|\Delta\nu_x| + \epsilon_x/4\right)\right\}^{\frac{1}{2}} \\ \epsilon_x &= \left|q \pm 2|\Delta\nu_x| - \nu_{x0} - \nu_{y0}\right| \end{aligned} \tag{4-10}$$

where ϵ_x is the distance from ν_{x0}, ν_{y0} to the stopband edge. One uses the + sign for the upper stopband edge and the − sign for the lower edge.

Close to the stopband edge, where $\epsilon_x \ll |\Delta\nu_x|$ then Eq. (4-10) gives the result

$$\left|\nu_x - \frac{1}{2}\left(\nu_{x0} + q - \nu_{y0}\right)\right| = \left\{\epsilon_x\,|\Delta\nu_x|\right\}^{1/2} \tag{4-11}$$

Equations (4-10) and (4-11) give the tune of the ν_x mode, ν_x, near the stopband edge. The result for the tune of the ν_y mode, ν_y, may be found by making the substitution $\nu_x \to \nu_y$, $\nu_{x0} \to \nu_{y0}$, $\nu_{y0} \to \nu_{x0}$, $|\Delta\nu_x| \to |\Delta\nu_y|$.

If one varies the unperturbed tune, ν_{x0}, ν_{y0}, so that the tune approaches the edge of the stopband, the tune on the stopband edge depends on the value of ν_{x0}, ν_{y0} when the unperturbed tune arrives at the stopband edge. The stopband edges are given by the two lines

$$\nu_{x0} + \nu_{y0} = q \pm 2|\Delta\nu|$$

where it is assumed that $|\Delta\nu_x| = |\Delta\nu_y| = |\Delta\nu|$ and the $+$ sign is for the upper edge and the $-$ sign for the lower edge.

The tune of the ν_x mode at the stopband edge is then given by

$$\begin{aligned} \nu_x &= \frac{1}{2}\left(\nu_{x0} + q - \nu_{y0}\right) \\ \nu_x &= \nu_{x0} \pm |\Delta\nu| \end{aligned} \tag{4-12}$$

where the $+$ sign is for the lower edge and the $-$ sign for the upper edge.

The tune of the ν_y mode at the stopband edge is given by

$$\nu_y = \nu_{y0} \pm |\Delta\nu|$$

One may note, that at the stopband edge

$$\begin{aligned} \nu_x + \nu_y &= \nu_{x0} + \nu_{y0} \pm 2|\Delta\nu| \\ \nu_x + \nu_y &= q \end{aligned} \tag{4-13}$$

and the ν_x, ν_y lies on the resonance line.

Eqs. (4-6) and (4-7) can also be rewritten as, for the ν_x mode and below the resonance line,

$$\begin{aligned} \nu_x &= \nu_{x0} + 0.5D\left\{1 - \left[\left(\frac{2\Delta\nu}{D}\right)^2\right]^{\frac{1}{2}}\right\} \\ D &= q - \nu_{x0} - \nu_{y0} \\ \Delta\nu &= \Delta\nu_x \simeq \Delta\nu_y \end{aligned} \tag{4-14a}$$

The equation for the ν_y mode is similar

$$\nu_y = \nu_{y0} + 0.5D\left\{1 - \left[\left(\frac{2\Delta\nu}{D}\right)^2\right]^{\frac{1}{2}}\right\} \tag{4-14b}$$

Using Eq. (4-14) one can compute how much ν_x, ν_y will move given the distance from the resonance, D, and the stopband width $\Delta\nu$. Eq. (4-14) show that as $\Delta\nu$ is increased, ν_x

and ν_y will move along the line from ν_{x0}, ν_{y0} which is perpendicular to the resonance line. When $\Delta\nu$ reaches 0.5 D, ν_x, ν_y will arrive on the resonance line where $\nu_x = \nu_{x0} + 0.5D$, $\nu_y = \nu_{y0} + 0.5D$, $\nu_x + \nu_y = q$.

5. The Beta Functions Near the Edge of a Stopband

In this paper it is being assumed that the unperturbed tune ν_{x0}, ν_{y0} is near the sum resonance $\nu_{x0} + \nu_{y0} = q$, q an integer, and the other linear resonances are far enough away so that the particle motion is dominated by the sum resonance. For this case, it will be shown that the beta functions β_x, β_y do not become infinite when ν_{x0}, ν_{y0} approach the edge of the stopband, as was found [2] for the case of uncoupled particle motion near a half integer resonance. If one goes to a coordinate system where the coordinates are uncoupled, then β_x for the uncoupled coordinates can become infinite, when the tune of the ν_x mode, ν_x is close to a half integer resonance, $\nu_x \simeq n/2$, n being an integer. It is assumed here that ν_{x0}, and thus ν_x is not near a half–integer resonance.

The beta functions, β_x, β_y, for linearly coupled motion may be defined by going to the coordinate system where the new coordinates u, p_u, v, p_v are uncoupled. If the u, p_u motion goes over into x, p_x motion, when the coupling goes to zero, then the beta function of the u, p_u motion will be called β_x. A similar definition is given to β_y.

In section 2, a solution was found for η_x, and $x = \beta_{x0}^{\frac{1}{2}} \eta_x$, when ν_{x0}, ν_{y0} are in the stable region near the edge of the stopband. β_x can be computed from this solution using the result, see section 6,

$$\frac{1}{\beta_x} = \frac{1}{\nu_{x0}\beta_{x0}} Im \frac{d}{d\theta_x} \log x \tag{5-1}$$

Im stand for the imaginary part and ν_{x0}, β_{x0} are the tune and beta function of the unperturbed motion. Eq. (5-1) holds for large accelerators. For small accelerators, where the large accelerator approximations are not used, it also requires that $B_x = 0$ on the closed orbit. Eq. (5-1) is derived in section 6.

The change in β_x due to the linear coupling field may be computed using Eq. (5-1) and the solution for $x = \beta_{x0}^{\frac{1}{2}}\eta_x$ found in section 3, Eq. (3-14),

$$
\begin{aligned}
\eta_x &= A_s \exp\left(i\nu_{x0}\theta_x\right)\left[1 + \sum_{m\neq 0} f_m \exp\left(-im\theta_x\right)\right] \\
f_m &= \frac{-2\nu_{x0}}{m\left(m - 2\nu_{x0}\right)} d_x b_m \exp\left(-i\delta_{1x}\right) \\
d_x &= \frac{-|\Delta\nu_x|}{\frac{1}{2}\left(q - \nu_{x0} - \nu_{y0}\right) + \delta_x} \\
b_m &= \frac{1}{4\pi}\int ds \frac{a_1}{\rho}\left(\beta_x\beta_y\right)^{1/2} \exp\left[-i\left(\left(q - \nu_{x0}\right)\theta_y + \nu_{x0}\theta_x\right) + im\theta_x\right] \\
\Delta\nu_x &= b_0, \quad \delta_{1x} = ph\left(\Delta\nu_x\right)
\end{aligned}
\tag{5-2}
$$

In the stable region, near a stopband edge, δ_x is given by

$$
\begin{aligned}
|\delta_x| &= \left\{\epsilon\left(|\Delta\nu_x| + \epsilon/4\right)\right\}^{\frac{1}{2}} \\
\epsilon &= |q + \pm|2\Delta\nu_x| - \nu_{x0} - \nu_{y0}|
\end{aligned}
\tag{5-3}
$$

with the $\pm$ sign for the upper and lower edge, respectively. ν_{xs} has been replaced by ν_x and δ_x is positive for the lower edge and negative for the upper edge. One then gets

$$
\begin{aligned}
Im\frac{d}{d\theta_x}\log x &= \nu_x + Im\sum_{m\neq 0}\left(-imf_m \exp\left(-im\theta_x\right)\right) \\
\frac{1}{\beta_x} &= \frac{1}{\nu_{x0}\beta_{x0}}\left(\nu_x - Re\sum_{m\neq 0} mf_m \exp\left(-im\theta_x\right)\right) \\
\frac{1}{\beta_x} &= \frac{1}{\beta_{x0}}\left(1 + \frac{\nu_x - \nu_{x0}}{\nu_{x0}} - \frac{1}{\nu_{x0}}Re\sum_{m\neq 0} mf_m \exp\left(-im\theta_x\right)\right) \\
\beta_x &= \beta_{x0}\left(1 - \frac{\nu_x - \nu_{x0}}{\nu_{x0}} + \frac{1}{\nu_{xo}}Re\sum_{m\neq 0} mf_m \exp\left(-im\theta_x\right)\right) \\
\frac{\beta_x - \beta_{x0}}{\beta_{x0}} &= -\frac{\nu_x - \nu_{x0}}{\nu_{x0}} + \frac{1}{\nu_{x0}}Re\sum_{m\neq 0} mf_m \exp\left(-im\theta_x\right)
\end{aligned}
\tag{5-4}
$$

The results for $(\beta_x - \beta_{x0})/\beta_{x0}$ is then

$$
\frac{\beta_x - \beta_{x0}}{\beta_{x0}} = -\frac{\nu_x - \nu_{x0}}{\nu_{x0}} - \sum_{m\neq 0}\frac{2}{m - 2\nu_{x0}} d_x Re\left(b_m \exp\left(-im\theta_x - i\delta_{1x}\right)\right) \tag{5-5}
$$

If one assumes that the harmonic $m \simeq 2\nu_{x0}$ dominate then the maximum change in β_x may be approximated by

$$\left|\frac{\beta_x - \beta_{x0}}{\beta_{x0}}\right|_{\max} = \left|\frac{\nu_x - \nu_{x0}}{\nu_{x0}}\right| + \frac{2|d_x||b_m|}{|m - 2\nu_{x0}|} \qquad (5\text{-}6)$$
$$m \simeq 2\nu_{x0}$$

ν_x is given by Eq. (4-10).

The results for β_y may be found by replacing each parameter with the corresponding parameter for the ν_y mode.

6. Small Accelerator Results

All the final results found in this paper will also hold for small accelerators where the exact equations of motion have to be used. The exact linear equations have the form [1]

$$\frac{dx_i}{ds} = \sum A_{ij} x_j, \qquad (6\text{-}1)$$

where i, j for from 1 to 4 and the x_i are the coordinates x, p_x, y, p_y. For large accelerators $p_x \simeq dx/ds$, $p_y \simeq dy/ds$, $A_{11} = A_{22} = A_{33} = A_{44} = 0$, and $A_{12} = A_{34} = 1$. The A_{ij} for the exact equations are given in reference 1. In particular

$$\begin{aligned}
&A_{12} = \frac{(1 + x/\rho)\left(1 - p_y^2\right)}{p_s^3}, \quad A_{34} = \frac{(1 + x/\rho)\left(1 - p_x^2\right)}{p_s^3} \\
&A_{13} = 0, \qquad A_{14} = \frac{(1 + x/\rho)\, p_x p_y}{p_s^3} \\
&p_s = \left\{1 - p_x^2 - p_y^2\right\}^{\frac{1}{2}}
\end{aligned} \qquad (6\text{-}2)$$

where the right hand side in the equation for A_{ij} are evaluated on the closed orbit.

The linear differential equations for η_x and η_y, $\eta_x = x/\beta_{x0}^{\frac{1}{2}}$, $\eta_y = \nu/\beta_{y0}^{\frac{1}{2}}$ can be found [1] as

$$\begin{aligned}
&\frac{d^2}{d\theta_x^2}\eta_x + \nu_{x0}^2 \eta_x = f_x \\
&\frac{d^2}{d\theta_y^2}\eta_y + \nu_{y0}^2 \eta_y = f_y \\
&f_x = \frac{\nu_{x0}^2 \beta_{x0}^{3/2}}{A_{12}} (1 + x/\rho)\, \Delta B_y \\
&f_y = -\frac{\nu_{yo}^2 \beta_{y0}^{3/2}}{A_{34}} (1 + x/\rho)\, \Delta B_x \\
&d\theta_x = A_{12} ds/\nu_{x0}\beta_{x0}, \quad d\theta_y = A_{34} ds/\nu_{y0}\beta_{y0}
\end{aligned} \qquad (6\text{-}3)$$

ΔB_x, ΔB_y are the perturbing fields given by Eqs. (2-1). For small accelerators, in order for Eqs. (6-3) to be valid, one also requires that the perturbing fields do not shift the closed orbit, or $\Delta B_x = \Delta B_y = 0$ on the closed orbit. If the closed orbit is shifted by the perturbing field, then the non–linear kinematic terms, the terms which do not explicitly depend on the field, will generate additional linear terms. ν_{x0}, ν_{y0}, β_{x0}, β_{y0} are the tune and beta functions of the unperturbed accelerator.

Comparing Eqs. (6-3) with the corresponding equations for large accelerators, Eqs. (2-2) one notes that f_x and f_y for the small accelerator have the additional factors of $1/A_{12}$ and $1/A_{34}$ respectively. Although the perturbation terms in Eqs. (6-3) now have the extra factors $1/A_{12}$ and $1/A_{34}$, these factors disappear in the final results when in the relevant integrals one goes from the variables θ_x or θ_y to the variable s according to Eqs. (6-3). Using Eqs. (6-3) one can go through the derivations and show that the final results for the tune, growth rates and beta functions are valid for both large and small accelerators.

One thing that remains to be done is to derive Eq. (5-1),

$$\frac{1}{\beta_x} = \frac{1}{\nu_{x0}\beta_{x0}} Im \frac{d}{d\theta_x} ln\ x \tag{6-4}$$

which allows β_x to be computed from the solutions for η_x, η_y.

The beta functions β_x, β_y for linearly coupled motion may be defined by going to the coordinate system where the near coordinates μ, p_u, v, p_v are uncoupled. If the μ, p_u motion goes into the x, p_x motion when the coupling goes to zero, then the beta function of the uncoupled μ, p_u motion will be called β_x. The solution of the equations of motion for μ, p_u may be related to β_x by [1]

$$\begin{aligned} \mu &= C\beta_2^{\frac{1}{2}} \exp(i\psi_x) \\ p_u &= C\beta_x^{-\frac{1}{2}} \exp(-\alpha_x + i) \exp(i\psi_x) \end{aligned} \tag{6-5}$$

C is a normalization constant. x, p_x and μ, p_u are related by the decoupling matrix, R [3]

$$\begin{aligned} \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} &= R \begin{pmatrix} u \\ p_u \\ v \\ p_v \end{pmatrix} \\ R &= \begin{pmatrix} \cos\phi I & D\sin\phi \\ -D^{-1}\sin\phi & \cos\phi I \end{pmatrix} \end{aligned} \tag{6-6}$$

I is the 2×2 identity matrix and D is a 2×2 matrix and $|D| = 1$. One then finds

$$\begin{aligned} x &= C\beta_x^{\frac{1}{2}} \exp(i\psi_x) \\ p_x &= C\beta_x^{-\frac{1}{2}} \exp(i\psi_x) \end{aligned} \tag{6-7}$$

From Eq. (6-7) one can relate β_x to x, p_x solutions by

$$\frac{1}{\beta_x} = Im\,(p_x/x) \tag{6-8}$$

where Im is the imaginary part. p_x may be eliminated by using Eq. (6-1)

$$p_x = \frac{1}{A_{12}}\left(\frac{dx}{ds} - A_{11}x - A_{13}y - A_{14}p_y\right)$$

which gives

$$\frac{1}{\beta} = Im\left[\frac{1}{x}\frac{1}{A_{12}}\left(\frac{dx}{ds} - A_{11}x - A_{13}y - A_{14}p_y\right)\right] \tag{6-9}$$

In the large accelerator approximation, $A_{12} = 1$ and $A_{13} = A_{14} = 0$ and (6-9) gives

$$\frac{1}{\beta} = \frac{Im}{\nu_{x0}\beta_{x0}}\frac{d}{d\theta_x}\log x \tag{6-10}$$

For small accelerators, one has (see reference 1)

$$\begin{aligned} A_{13} &= \frac{\partial}{\partial y}\left[\frac{(1+x/\rho)\,p_x}{p_s}\right] = 0 \\ A_{14} &= \frac{\partial}{\partial p_y}\left[\frac{(1+x/\rho)\,p_x}{p_s}\right] = (1+x/\rho)\,\frac{p_x p_y}{p_s^3} \\ p_s &= \left[-1 - p_x^2 - p_y^2\right]^{1/2} \end{aligned} \tag{6-11}$$

Thus if $\Delta B_x = 0$ on the closed orbit, so that $p_y = 0$ on the closed orbit then $A_{14} = 0$ for small accelerators too. Since $A_{12}ds = \nu_{x0}\beta_{x0}d\theta_x$, one gets Eq. (6-10) again for small accelerators provided $\Delta B_x = 0$ on the unperturbed closed orbit.

7. Comments on the Results

Others have worked on this subject and there is an overlap between the contents of this paper and their work. These previous papers (4 to 12) give results for the stopband width and for the growth rate inside the stopband.

The results in this paper include the following:

1. Results for the tune in the stable region near an edge of the stopband. The results show that as ν_{x0}, ν_{y0} approach the edge of the stopband, the tunes of the two normal modes ν_x and ν_y begin to change rapidly and when ν_{x0}, ν_{y0} reach the stopband edge then ν_x and ν_y lie on the resonance line $\nu_x + \nu_y = q$. These final values of ν_x, ν_y, when ν_{x0}, ν_{y0} reach the stopband edge, are approached like $\epsilon^{\frac{1}{2}}$, where ϵ is the distance from ν_{x0}, ν_{y0} to the stopband edge.
2. Results for the beta functions of the normal modes, β_x, β_y, in the stable region near the edge of a stopband. The results show that β_x, β_y do not become infinite when ν_{x0}, ν_{y0} approach the stopband edge, unless ν_{x0}, ν_{y0} are near the half integer resonances $v_x = m/2$, or $v_y = n/2$, m and n being integers.
3. Results for the 2 solutions of the equations of motion in the stable region near a stopband edge and in the unstable region.
4. The above results hold also for small accelerators, where the exact equations of motion have to be used and the large accelerator approximation is not valid. For small accelerators, one needs the restriction that the perturbing field gradients do not shift the closed orbit.

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